Automorphisms of 3-dimensional Abelian Varieties

Ch. Birkenhake, V. González-Aguilera, H. Lange

Abstract. We give a classification of automorphisms of 3-dimensional abelian varieties, completing the results of Roan (see [**RO**]).

Introduction

A complex torus of dimension g is by definition a quotient $X = \mathbb{C}^g/\Lambda$ of the vector space $\mathbb{C}^g$ modulo a lattice Λ of maximal rank. X is called an *abelian variety* if it admits a *polarization*, that is a positive definite hermitian from H on $\mathbb{C}^g$ whose imaginary part is integer valued on Λ. According to the elementary divisor theorem Λ admits a basis such that the matrix of $\operatorname{Im} H$ is of the form $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ with a diagonal matrix $D = \operatorname{diag}(d_1, \cdots, d_g)$ and positive integers d_i with $d_i | d_{i+1}$ for $i = 1, \cdots, g$. The g-tupel $(d_1, \cdots, d_g)$ or the matrix D is called the *type* of the polarization H. A polarization of type $(1, \cdots, 1)$ is called a *principal polarization.* The pair (X, H) is called a *polarized abelian variety.* An *automorphism* of X is by definition a biholomorphic map $\alpha : X \to X$ which respects the group law of X. The automorphisms of X form a group. This group is denoted by $\operatorname{Aut}(X)$. In general it is an infinite group. For example if $X = E \times \cdots \times E$, the n-fold product of a general elliptic curve E with itself, then $\operatorname{Aut}(X) = GL_n(\mathbb{Z})$.

An automorphism α of $X = \mathbb{C}^g/\Lambda$ induces an automorphism of the vector space $\mathbb{C}^g$, the *analytic representation* $\rho_a(\alpha)$ of α. We say that α *respects the polarization* H of X if $\rho_a(\alpha)^* H = H$. In that case α is called an *automorphism of the polarized abelian variety* (X, H). The automorphisms of (X, H) form a group denoted by $\operatorname{Aut}_H(X)$, which is always a finite group [**CAV**]. It is well known that for an elliptic curve E and any polarizarion H on E we have $\operatorname{Aut}(E) = \operatorname{Aut}_H(E) = \mathbb{Z}_d$, the cyclic group of order d with $d = 2, 4$ or 6.

Much is known about automorphisms as well as the groups $\operatorname{Aut}(S)$ and $\operatorname{Aut}_H(S)$ in the case of an abelian surface S (see [**FU**]). In [**BGL**] we study the group $\operatorname{Aut}(X)$ of an *abelian threefold* X, that is an abelian variety X of dimension 3. To be more precise we determine the maximal finite subgroups of $\operatorname{Aut}(X)$ in this case or, slightly more generally, for any complex torus of dimension 3. It is the aim of this note to present some results on automorphisms as well as on the group $\operatorname{Aut}_H(X)$ in the case of polarized abelian threefolds (X, H). In order to be as complete as possible we also include some apparently unpublished results of Roan [**RO**], of the

---

1991 *Mathematics Subject Classification.* Primary: 14K10; Secondary: 14K99.
Supported by Fondecyt 8970007, DGIP U.Santa María and DAAD.

thesis of D. Schmidt [**SCH**] (written under the supervision of the first and third author), and other authors, sometimes without giving explicit attributions.

There are several reasons why one is interested in automorphisms of polarized abelian threefolds:

(1) As for every geometric structure the group of automorphisms is an important invariant. An ultimate goal would be to give a stratification of the moduli space $\mathcal{A}_D$ of polarized abelian varieties of type D.
(2) For any pair (X, H) as above there is an ample line bundle L with first Chern class H. Any two such line bundles differ by a translation of X. Let $\varphi_L : X \to \mathbb{P}_N = P(H^0(L)^*)$ denote the rational map defined by the complete linear system $|L|$. It is often hard to study geometric properties of the map φ_L, in particular in the threefold case. Sometimes one can use automorphisms for these questions. See for example [**BL**].
(3) If C is a smooth projective curve of genus 3 over the field of complex numbers, its Jacobian $(J(C), \Theta)$ is a principally polarized abelian threefold. Torelli's theorem implies that there is a close relationship between the groups $\mathrm{Aut}\,(C)$ and $\mathrm{Aut}_{\Theta}(J(C))$.
   (i) In some cases one can use the automorphisms to determine an explicit period matrix for $J(C)$ (see e.g. [**RAU**]).
   (ii) One can use the automorphism groups to determine a splitting of $J(C)$ (see e.g. [**RIE**]).
(4) The desingularization of the quotient of an abelian threefold modulo a finite group of automorphisms may lead to interesting threefolds. In some cases these varieties are known to be Calabi-Yau threefolds. One should determine all groups of automorphisms which lead to Calabi-Yau threefolds in this way.

This paper is organized as follows: In Section 1 we recall some versions of the fixed point formula and prove some preliminary results. Section 2 contains the decomposition theorem. In Section 3 we study abelian threefolds of CM-type. Section 4 contains the classification of polarized abelian threefolds with an automorphism α of order d such that $X_{<\alpha>} := \cup_{k=1}^{d-1} \mathrm{Fix}\,(\alpha^k)$ is finite. Finally in Section 5 we study abelian threefolds X with an automorphism α such that $\dim X_{<\alpha>} \geq 1$.

1. The Fixed Point Formula

Let $X = \mathbb{C}^g/\Lambda$ be an abelian variety of dimension g. Denote by $\mathrm{End}\,(X)$ the ring of endomorphisms $f : X \to X$. In these terms the group $\mathrm{Aut}\,(X)$ is just the groups of units in $\mathrm{End}\,(X)$. For any integer n different from zero we denote by $n_X : X \to X$ the endomorphism defined by $n_X(x) = nx$. Its kernel X_n is called the group of n-division points.

Let $f \in \mathrm{End}\,(X)$ be any endomorphism. Denote by $\mathrm{Fix}\, f$ the subset of X of fixed points of f:

$$\mathrm{Fix}\, f := \ker\,(\mathrm{id}_X - f)$$

Clearly $\mathrm{Fix}\, f$ is a subgroup either of positive dimension or finite. Denote $\#\mathrm{Fix}\, f := \deg(\mathrm{id}_X - f)$, i.e.

$$\#\mathrm{Fix}\, f = \begin{cases} \text{cardinality of } \ker(\mathrm{id}_X - f) & \text{if } \dim \mathrm{Fix}\, f = 0 \\ 0 & \phantom{\text{if }} \dim \mathrm{Fix}\, f > 0 \end{cases}.$$

In order to state the Fixed Point Formula we first need to introduce the analytic and rational representation of End(X): Any endomorphism $f \in \mathrm{End}(X)$ lifts to a $\mathbb{C}$-linear map $\rho_a(f) : \mathbb{C}^g \to \mathbb{C}^g$ in a unique way. Choose an isomorphism $\varphi : \mathbb{Z}^{2g} \to \Lambda$. Then the restriction of $\rho_a(f)$ to Λ defines a $\mathbb{Z}$-linear map

$$\rho_r(f) := \varphi^{-1} \cdot (\rho_a(f)|\Lambda) \cdot \varphi : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}.$$

The assignments $f \mapsto \rho_a(f)$ and $f \mapsto \rho_r(f)$ define representations

$$\rho_a : \mathrm{End}(X) \to GL_g(\mathbb{C})$$

and

$$\rho_r : \mathrm{End}(X) \to GL_{2g}(\mathbb{Z})$$

called the *analytic* and the *rational representation* of End(X). By construction $\rho_a(f)$ and $\rho_r(f)$ fit into the following diagram of short exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{2g} & \xrightarrow{\varphi} & \mathbb{C}^g & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_r(f)} & & \downarrow{\scriptstyle \rho_a(f)} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \mathbb{Z}^{2g} & \xrightarrow{\varphi} & \mathbb{C}^g & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Using this notation we have

LEMMA 1.1 (Fixed Point Formula, 1. Version). a) Fix f *is a closed subgroup of* X *of dimension equal to the multiplicty of* 1 *as an eigenvalue of* $\rho_a(f)$.
b) $\#\mathrm{Fix}\, f = \det(\mathbf{1}_{2g} - \rho_r(f)) = |\det(\mathbf{1}_g - \rho_a(f))|^2$

PROOF. The first statement is obvious. For b) we recall that the rational representation $\rho_r(g)$ of an endomorphism g of X is equivalent (over $\mathbb{C}$) to the direct sum of representations $\rho_a(g) \oplus \overline{\rho_a(g)}$ (see [**CAV**], Section 1.2). Moreover the above diagram shows that

$$\ker g = \ker(\rho_a(g))/\varphi(\ker(\rho_r(g))) = \ker(\rho_a(g))/\Lambda \cap \ker(\rho_a(g)).$$

Hence

$$\deg g = \det \rho_r(g) = |\det(\rho_a(g))|^2 .$$

Now apply this equation to $g = \mathrm{id}_X - f$. □

There is a more subtle version of for Fixed Point Formula which involves the special structure of the endomorphism algebra of X: According to Poincare's Reducibility Theorem there is an isogeny

$$X \to X_1^{n_1} \times X_2^{n_2} \times \cdots \times X_r^{n_r}$$

with simple and pairwise non-isogenous abelian varieties X_i of dimension g_i. Denote by

$$\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the endomorphism algebra of $\mathrm{End}_{\mathbb{Q}}(X)$. Then the above isogeny induces a $\mathbb{Q}$-algebra isomorphism

$$(*) \qquad \mathrm{End}_{\mathbb{Q}}(X) \approx \oplus_{i=1}^r M_{n_i}(\mathrm{End}_{\mathbb{Q}}(X_i)).$$

Recall that the endomorphism algebra of a simple abelian variety is a skew-field of finite dimension over $\mathbb{Q}$ (see [**CAV**]). Denote $D_i = \mathrm{End}_{\mathbb{Q}}(X_i)$, $K_i := \mathrm{center}(D_i)$, $e_i := [K_i : \mathbb{Q}]$, $d_i^2 := [D_i : K_i]$ and $N_i : D_i \to \mathbb{Q}$ the reduced norm map. Note that

N_i is well defined even in the non-commutative case. Moreover for $f \in \operatorname{End}_{\mathbb{Q}}(X)$ let

$$f = f_1 + \cdots + f_r$$

be the decomposition (*), i.e., $f_i \in M_{n_i}(\operatorname{End}_{\mathbb{Q}}(X_i))$. With this notation we have

LEMMA 1.2 (Fixed Point Formula). *The number of fixed points of* $f = \sum_{i=1}^{r} f_i \in \operatorname{End}_{\mathbb{Q}}(X) = \oplus_{i=1}^{r} M_{n_i}(\operatorname{End}_{\mathbb{Q}}(X_i)))$ *is*

$$\#\operatorname{Fix} f = \prod_{i=1}^{r} N_i(\det(1 - f_i))^{\frac{2g_i}{d_i e_i}}.$$

PROOF. In view of the first version of the Fixed Point Formula and the decomposition (*) it suffices to show that the restriction of the map $\det \circ \rho_r : \operatorname{End}_{\mathbb{Q}}(X) \to \mathbb{Q}$ to $\operatorname{End}_{\mathbb{Q}}(X_i^{n_i}) = M_{n_i}(D_i)$ is $N_i(\det(\cdot))^{\frac{2g_i}{d_i e_i}}$. To see this note that $\det \circ \rho_r$ is of degree $2g_i n_i$ on $\operatorname{End}_{\mathbb{Q}}(X_i^{n_i})$, X_i being of dimension g_i. On the other hand $N_i \circ \det : M_{n_i}(D_i) \to \mathbb{Q}$ is of degree $n_i d_i e_i$. But any two norm maps of the same degree coincide (see [**MUM**] Lemma p. 179), we get $(\det \circ \rho_r)^{d_i e_i} = (N_i \cdot \det)^{2g_i}$. This implies the assertion. □

REMARK 1.3. Although the components f_i of f depend on the choice of the isogeny $X \to \prod_{i=1}^{n} X_i^{n_i}$, the norm $N_i(\det(1 - f_i))$ does not.

Next we apply the Fixed Point Formula to automorphisms and groups of automorphisms of the abelian variety X.

First let us fix some notation: Suppose $G \subset \operatorname{Aut}(X)$ is a finite group of automorphisms. Denote

$$X_G := \bigcup_{\operatorname{id}_X \neq \alpha \in G} \operatorname{Fix} \alpha.$$

According to Lemma 1 X_G is a closed algebraic subset of X. It was introduced by Roan in [**RO**]. Given an automorphism α denote by d_α its order. An immediate consequence of the Fixed Point Formula is:

PROPOSITION 1.4. *X_G is finite if and only if for all $\operatorname{id}_X \neq \alpha \in G$ all eigenvalues of $\rho_a(\alpha)$ (or equivalently those of $\rho_r(\alpha)$) are primitive d_α-th roots of unity.*

REMARK 1.5. In the case of an abelian group it suffices to check the if condition of the Proposition for any set of generators of G. In particular, given a cyclic group $G = \langle \alpha \rangle$ of order d the set X_G is finite if and only if all eigenvalues of $\rho_a(\alpha)$ are primitive d-th roots of unity.

Suppose now that α is an automorphism of X of order d and let $G = \langle \alpha \rangle$ be the associated cyclic group. Assume X_G is finite. Then the fixed points of α are division points. To be more precise we have

PROPOSITION 1.6. *Suppose $X_{<\alpha>}$ is finite. Then*

$$\operatorname{Fix} \alpha \subset \bigcap_{n | d_n \geq 2} X_n.$$

PROOF. All eigenvalues of $\rho_a(\alpha)$ are primitive d-th roots of unity, X_G being finite. Let $n \geq 2$ be a divisor of d. If $\beta = \alpha^{\frac{d}{n}}$, all eigenvalues of $\rho_a(\beta)$ are primitive n-th roots of unity. Hence $\sum_{i=0}^{n-1} \beta^i = 0$ and for any $x \in \operatorname{Fix}\beta$ we have

$$nx = nx - \left(\sum_{i=0}^{n-1} \beta^i\right)(x) = nx - \sum_{i=0}^{n-1} x = 0.$$

Now the assertion follows from the fact that $\operatorname{Fix}\alpha \subseteq \operatorname{Fix}\beta$. □

COROLLARY 1.7. *$\#\operatorname{Fix}\alpha \geq 2$ implies that $d = p^k$ for some prime number p. Moreover $\operatorname{Fix}\alpha \subset X_p$ in this case.*

Denote by φ the Euler function counting the number of primitive roots of unity of a given order d. Then we have

PROPOSITION 1.8. *Suppose α is an automorphism of order d of the g-dimensional abelian variety X such that $X_{\langle\alpha\rangle}$ is finite, then*

a) *$\varphi(d)$ divides $2g$.*
b) *The set $\Phi_\alpha := \{$eigenvalues of $\rho_a(\alpha)\}$ contains $\frac{\varphi(d)}{2}$ different pairwise non complex conjugate primitive d-th roots of unity.*
c) *If $d = p^k$ with p prime, $\#\operatorname{Fix}\alpha = p^{\frac{2g}{\varphi(d)}}$.*

PROOF. According to Proposition 1.4 all eigenvalues of $\rho_r(\alpha)$ are primitive d-th roots of unity. Thus the characteristic polynomial of $\rho_r(\alpha)$,

$$P^r_\alpha(\lambda) := \det(\lambda \mathbf{1}_{2g} - \rho_r(\alpha)),$$

is a multiple of the d-th cyclotomic polynomial μ_d. Hence $\varphi(d) = \deg \mu_d$ must divide $\deg P^r_\alpha = 2g$. This proves a).
The set $\Phi_\alpha \cup \overline{\Phi_\alpha}$ is the set of all eigenvalues of the rational representation $\rho_r(\alpha)$, since ρ_r and $\rho_a \oplus \overline{\rho}_a$ are equivalent representations. But all $\varphi(d)$ d-th roots of unity occur as an eigenvalues of $\rho_r(\alpha)$, since $\mu_d | P^r_\alpha$. This implies b). Comparing degrees we have $P^r_\alpha = \mu_d^{\frac{2g}{\varphi(d)}}$. Hence by the Fixed Point Formula

$$\#\operatorname{Fix}\alpha = \det(1_{2g} - \rho_r(\alpha)) = P^r_\alpha(1) = \mu_\alpha(1)^{\frac{2g}{\varphi(d)}}.$$

Now c) follows from the well known fact that for $d = p^k$:

$$\mu_\alpha(1) = N_{\mathbb{Q}(\xi_d)/\mathbb{Q}}(1 - \xi_d) = p$$

where ξ_d is a primitive d-th root of unity and $N_{\mathbb{Q}(\xi_d)/\mathbb{Q}}$ is the norm map. □

EXAMPLE 1.9. Let α be an automorphism of an elliptic curve of order $d > 1$. Then we have the following possibilities

d	2	3	4	6
$\#\operatorname{Fix}\alpha$	4	3	2	1

PROOF. According to Proposition 1.8 a) the Euler number of d satisfies $\varphi(d)|2$, hence $d \in \{2, 3, 4, 6\}$ and Proposition 1.8 c) and Corollary 1.7 give the assertion. □

It is a well-known fact that these automorphisms can be realized by the elliptic curves with j-invariant 0 and 1728 .

EXAMPLE 1.10. Let α be an automorphism of an abelian surface with $X_{\langle\alpha\rangle}$ finite. Then

d	2	3	4	5	6	8	10	12
$\#\mathrm{Fix}\,\alpha$	16	9	4	5	1	2	1	1

The proof is again an easy computation using Propositions 1.8 and 1.4. □

2. The Decomposition Theorem

Suppose X is an abelian variety of dimension g and α is an automorphism of X of order d. Denote by $G = \langle\alpha\rangle$ the cyclic group defined by α. In this section we study the case that the set X_G is of positive dimension. We will see that G defines a decomposition of X up to isogeny into subtori of X invariant with respect to the action of G. To be more precise we show the following theorem due to Roan.

THEOREM 2.1 (Decomposition Theorem). *Let $G = \langle\alpha\rangle$ be a cyclic group of automorphisms of X. Suppose $1 \leq d_1 < d_2 < \ldots < d_r$ are the orders of the eigenvalues of $\rho_a(\alpha)$. Then there are G-stable abelian subvarieties $X_1, \cdots, X_r$ of X such that*

a) $\alpha_i = \alpha|X_i$ *is of order* d_i

b) $(X_1)_{\langle\alpha_1\rangle} = \begin{cases} X_1 & \text{if } d_1 = 1 \\ \text{finite} & \text{if } d_1 > 1 \end{cases}$ *and* $(X_i)_{\langle\alpha_i\rangle}$ *is finite for* $i > 1$.

c) *The addition map*

$$\mu : X_1 \times X_2 \times \cdots \times X_r \to X$$

is an isogeny.

REMARK 2.2. One can describe the subtori X_i in the Decomposition Theorem explicitly: Let the notation be as in the Theorem. Define a filtration of X into G-stable abelian subvarieties

$$0 = Y_r \subset Y_{r-1} \subset \cdots \subset Y_1 \subset Y_0 = X$$

by

$$Y_i := \mathrm{im}\left(\mathrm{id} - \alpha^{d_i}\right)\big| Y_{i-1}.$$

Then

$$X_i := \ker\left((\mathrm{id} - \alpha^{d_i})\big| Y_{i-1}\right)_0 = \mathrm{Fix}\left(\alpha^{d_i}\big| Y_{i-1}\right)_0$$

where the index 0 stands for the component of the kernel containing 0.

PROOF OF THE DECOMPOSITION THEOREM. Let the notation be as in the Remark. Obviously the abelian subvarieties Y_i as well as X_i are G-stable. By construction all the eigenvalues of $\rho_a(\alpha|X_i)$ are primitive of order d_i, this implies a) and b).

As for c): Denote for abbreviation $f_i := (\mathrm{id} - \alpha^{d_i})|Y_{i-1}$ for $i = 1, \cdots, r$. So f_i is an endomorphism of Y_{i-1} and $Y_i = \mathrm{im} f_i$ and $X_i = (\ker f_i)_0$ are abelian subvarieties of Y_{i-1}. In particular this shows that the addition map $X_i \times Y_i \to Y_{i-1}$ is an isogeny, for $i = 1, \cdots, r$. Putting all these addition maps together we get a sequences of isogenies

$$Y_r \times X_r \times X_{r-1} \times \cdots \times X_1 \to \cdots \to Y_2 \times X_2 \times X_1 \to Y_1 \times X_1 \to Y_0 = X.$$

This completes the proof of c), since $Y_r = 0$. □

PROPOSITION 2.3. *Suppose $G \subset \operatorname{Aut} X$ is a finite abelian group of automorphisms of an abelian variety X such that X_G is finite. Then G is a cyclic group.*

PROOF. The analytic representation induces a linear action of G on the $\mathbb{Q}$-vector space $H_1(X, \mathbb{Q})$. Let

$$H_1(X, \mathbb{Q}) = V_1 \oplus \cdots \oplus V_s$$

be its decomposition into irreducible G-modules and denote by G_i the image of G in $GL(V_i)$. Since G and thus also G_i is abelian, the group G_i is even contained in the center of $GL_{G_i}(V_i)$. But $GL_{G_i} V_i$ is the group of units of $\operatorname{End}_{G_i} V_i$, which is a skew field by Schur's Lemma. So G_i is cyclic, being a finite subgroup of the group of units of a skew field.

We have to show that G is cyclic too. To see this consider the restriction map: $\operatorname{res}_i : G \to G_i$. By construction res_i is onto. Suppose $\alpha \in \ker \operatorname{res}_i \subset G$, i. e., $\alpha | V_i$ is the identity. By Proposition 1.4 $\alpha = \operatorname{id}_X$, so res_i is an isomorphism. This completes the proof. □

3. Abelian Threefolds of CM-type

In this section we study some abelian threefolds of CM-type with automorphism α of order 7 and 9. We give the period matrices, the rational and the analytic representations of α in some normalized form. Moreover we prove a theorem due to Roan, which is essential for the classification of abelian threefolds with automorphism of the next section.

A *CM-field* K is by definition a totally imaginary quadratic extension of a totally real number field, say of degree g over $\mathbb{Q}$. A CM-type of K is a set $\Phi = \{\sigma_1, \cdots, \sigma_g\}$ of embeddings $K \mapsto \mathbb{C}$, pairwise not complex conjugate. An abelian variety $X = \mathbb{C}^g/\Lambda$ is called to be of *CM-type* (K, Φ), if there is an embedding $\rho : K \mapsto \operatorname{End}_{\mathbb{Q}}(X)$, such that:

$$\rho_a \circ \rho \sim_{\mathbb{C}} \operatorname{diag}(\sigma_1, \cdots, \sigma_g) : K \mapsto M_g(\mathbb{C}).$$

Here "$\sim_{\mathbb{C}}$" means equivalence of representations over $\mathbb{C}$.

To every CM-type Φ of K one can associate an abelian variety $X(K, \Phi)$ in a canonical way: The tensor product $K \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-vector space of dimension $2g$. The CM-type $\Phi = (\sigma_1, \cdots, \sigma_g)$ induces a complex structure on $K \otimes_{\mathbb{Q}} \mathbb{R}$ via the $\mathbb{R}$-linear isomorphism

$$(\sigma_1, \cdots, \sigma_g) : K \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g.$$

The ring of integers $\mathcal{O}$ of K is a lattice of rank $2g$ in $K \otimes_{\mathbb{Q}} \mathbb{R}$. Hence the quotient

$$X(K, \Phi) := K \otimes_{\mathbb{Q}} \mathbb{R}/\mathcal{O}$$

is a complex torus of dimension g. According to [**MUM**] p. 213 $X(K, \Phi)$ is an abelian variety. By construction we have

$$\mathcal{O} \subset \operatorname{End}(X(K, \Phi))$$

implying

$$\mathcal{O}^* \subset \operatorname{Aut}(X(K, \Phi)).$$

There are exactly two cyclotomic fields leading to abelian threefolds in this way, namely $\mathbb{Q}(\xi_7)$ and $\mathbb{Q}(\xi_9)$. The cyclotomic field $\mathbb{Q}(\xi_7)$ admits 2 different CM-types, namely $\Phi_1^7 = \{\xi_7, \xi_7^2, \xi_7^3\}$ and $\Phi_2^7 = \{\xi_7, \xi_7^2, \xi_7^4\}$. Similarly $\mathbb{Q}(\xi_9)$ admits two different CM-types $\Phi_1^9 = \{\xi_9, \xi_9^2, \xi_9^4\}$ and $\Phi_2^9 = \{\xi_9, \xi_9^4, \xi_9^7\}$.

PROPOSITION 3.1. *There exist 4 abelian threefolds $X(\mathbb{Q}(\xi_d),\Phi_i^d)$, for $d=7$ and $d=9$ and $i=1,2$, admiting automorphisms ξ_7 and $-\xi_7$ (respectively ξ_9 and $-\xi_9$) of order 7 and 14 (respectively of order 9 and 18) on $X(\mathbb{Q}(\xi_7),\Phi_i^7)$, (respectively $X(\mathbb{Q}(\xi_9),\Phi_i^9)$). The matrices Π_i^d below are period matrices for $X(\mathbb{Q}(\xi_d),\Phi_i^d)$:*

$$\Pi_1^7=\begin{pmatrix} -1+2\xi_7+\xi_7^3+\xi_7^6 & 1+\xi_7^3+\xi_7^5 & 1+\xi_7^5-\xi_7^6 & 1\ 0\ 0 \\ 1+\xi_7^3+\xi_7^5 & \xi_7^2 & -\xi_7-\xi_7^3-\xi_7^5 & 0\ 1\ 0 \\ 1+\xi_7^5-\xi_7^6 & -\xi_7-\xi_7^3-\xi_7^5 & -\xi_7^5 & 0\ 0\ 1 \end{pmatrix}$$

$$\Pi_2^7=\begin{pmatrix} 3\frac{\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-2+\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-4-\xi_7-\xi_7^2-\xi_7^4}{4} & 1\ 0\ 0 \\ \frac{-2+\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-2+\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-10+\xi_7+\xi_7^2+\xi_7^4}{4} & 0\ 1\ 0 \\ \frac{-4-\xi_7-\xi_7^2-\xi_7^4}{4} & \frac{-10+\xi_7+\xi_7^2+\xi_7^4}{4} & 7\frac{-4+\xi_7+\xi_7^2+\xi_7^4}{4} & 0\ 0\ 7 \end{pmatrix}$$

$$\Pi_1^9=\begin{pmatrix} \begin{matrix}\frac{1}{397}(174+139\xi_9\\ +54\xi_9^2+135\xi_9^3\\ +101\xi_9^4-95\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(-515-3\xi_9\\ -64\xi_9^2-160\xi_9^3\\ -105\xi_9^4+142\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(-549-199\xi_9\\ -143\xi_9^2-159\xi_9^3\\ +181\xi_9^4-241\xi_9^5)\end{matrix} & 1\ 0\ 0 \\ \begin{matrix}\frac{1}{397}(-515-3\xi_9\\ -64\xi_9^2-160\xi_9^3\\ -105\xi_9^4+142\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(1125+180\xi_9\\ +267\xi_9^2+72\xi_9^3\\ +345\xi_9^4-183\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(783+30\xi_9\\ +243\xi_9^2+12\xi_9^3\\ -141\xi_9^4+168\xi_9^5)\end{matrix} & 0\ 3\ 0 \\ \begin{matrix}\frac{1}{397}(-549-199\xi_9\\ -143\xi_9^2-159\xi_9^3\\ +181\xi_9^4-241\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(783+30\xi_9\\ +243\xi_9^2+12\xi_9^3\\ -141\xi_9^4+168\xi_9^5)\end{matrix} & \begin{matrix}\frac{1}{397}(1917+402\xi_9\\ +636\xi_9^2+399\xi_9^3\\ -222\xi_9^4+822\xi_9^5)\end{matrix} & 0\ 0\ 3 \end{pmatrix}$$

$$\Pi_2^9=\begin{pmatrix} 1+\xi_9^3 & 0 & 0 & 1\ 0\ 0 \\ 0 & 1+\xi_9^3 & 0 & 0\ 1\ 0 \\ 0 & 0 & 1+\xi_9^3 & 0\ 0\ 1 \end{pmatrix}$$

Π_1^7 and Π_2^9 define principal polarizations, Π_2^7 defines a polarization of type $(1,1,7)$ and Π_1^9 defines a polarization of type $(1,3,3)$.

PROOF. Theorem 4.2 below implies that $X(\mathbb{Q}(\xi_d),\Phi_i^d)$ are the only abelian threefolds on which ξ_d induces an automorphism of order d for $d=7$ and $d=9$. Hence it suffices to show that there are matrices $\rho_a^i(\xi_d)\in M_3(\mathbb{C})$ and $\rho_r^i(\xi_d)\in M_6(\mathbb{Z})$ of order d such that

$$(*)\qquad \rho_a^i(\xi_d)\Pi_i^d=\Pi_i^d\rho_r^i(\xi_d)$$

These matrices are then the analytic and rational representation of the automorphism ξ_d. It is an immediate computation to check equation (*) with the following pairs of matrices:

$$\rho_a^1(\xi_7)=\begin{pmatrix} -1+2\xi_7+\xi_7^3+\xi_7^6 & -1+\xi_7-\xi_7^2+\xi_7^3-\xi_7^4 & 1+2\xi_7+2\xi_7^3+2\xi_7^5 \\ 1+\xi_7^3+\xi_7^5 & 1+\xi_7^2+\xi_7^3+\xi_7^5 & 1-\xi_7+\xi_7^2 \\ 1+\xi_7^5-\xi_7^6 & 2+\xi_7^2+\xi_7^4+\xi_7^5 & 1+\xi_7^2+\xi_7^4 \end{pmatrix}$$

and

$$\rho_r^1(\xi_7)=\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix},$$

$$\rho_a^2(\xi_7)=\begin{pmatrix} 1+3\frac{\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-2-\xi_7-\xi_7^2-\xi_7^4}{4} & \frac{5+3(\xi_7+\xi_7^2+\xi_7^4)}{4} \\ \frac{2+\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{-2-\xi_7-\xi_7^2-\xi_7^4}{4} & \frac{1+2(\xi_7+\xi_7^2+\xi_7^4)}{7} \\ -1-\frac{\xi_7+\xi_7^2+\xi_7^4}{4} & \frac{10-\xi_7-\xi_7^2-\xi_7^4}{4} & \frac{-1+\xi_7+\xi_7^2+\xi_7^4}{2} \end{pmatrix}$$

and

$$\rho_r^2(\xi_7) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -7 & 0 & -1 & 7 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & -3 & -7 & 1 & -1 & 7 \\ 0 & -2 & -7 & 1 & -1 & 7 \\ 0 & -1 & -4 & 0 & 0 & 3 \end{pmatrix},$$

$$\rho_a^1(\xi_9) = \begin{pmatrix} \begin{matrix} \frac{1}{397}(78+76\xi_9 \\ -99\xi_9^2-49\xi_9^3 \\ +278e_7^4-289e_7^5) \end{matrix} & \begin{matrix} \frac{1}{1191}(-381+209\xi_9 \\ -173\xi_9^2-234\xi_9^3 \\ +169\xi_9^4-100\xi_9^5) \end{matrix} & \begin{matrix} \frac{1}{1191}(642-199\xi_9 \\ -143\xi_9^2-159\xi_9^3 \\ +181\xi_9^4-241\xi_9^5) \end{matrix} \\ \begin{matrix} \frac{1}{397}(-313+204\xi_9 \\ +382\xi_9^2-236\xi_9^3 \\ -6\xi_9^4+269\xi_9^5) \end{matrix} & \begin{matrix} \frac{1}{397}(77+187\xi_9 \\ +284\xi_9^2-84\xi_9^3 \\ +193\xi_9^4+15\xi_9^5) \end{matrix} & \begin{matrix} \frac{1}{397}(-136+10\xi_9 \\ +81\xi_9^2+4\xi_9^3 \\ -47\xi_9^4+56\xi_9^5) \end{matrix} \\ \begin{matrix} \frac{1}{397}(411+34\xi_9 \\ +593\xi_9^2+93\xi_9^3 \\ -\xi_9^4+508\xi_9^5) \end{matrix} & \begin{matrix} \frac{1}{397}(79-35\xi_9 \\ +312\xi_9^2-14\xi_9^3 \\ -34\xi_9^4+201\xi_9^5) \end{matrix} & \begin{matrix} \frac{1}{397}(-155+134\xi_9 \\ +212\xi_9^2+133\xi_9^3 \\ -74\xi_9^4+274\xi_9^5) \end{matrix} \end{pmatrix}$$

and

$$\rho_r^1(\xi_9) = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 & 0 \\ -1 & 2 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -2 & 3 & 3 & 2 & 3 & 3 \\ 1 & -2 & -1 & -1 & -2 & -1 \\ 1 & -2 & -2 & -1 & -2 & -2 \end{pmatrix},$$

$$\rho_a^2(\xi_9) = \begin{pmatrix} 0 & 0 & \xi_9^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho_r^2(\xi_9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

□

Let X be an abelian variety of dimension g and α an automorphism of order d of X with $X_{\langle\alpha\rangle}$ finite. According to Proposition 1.8 $\varphi(d)$ divides $2g$. In [**BGL**], Proposition 3.3 we showed that if $\varphi(d) = 2g$, then $(\mathbb{Q}(\xi_d), \Phi_\alpha)$ is a CM-field and X is an abelian variety of CM-type $(\mathbb{Q}(\xi_d), \Phi_\alpha)$. The following theorem due to Roan (see [**RO**]) is a generalization of this.

THEOREM 3.2 (Roan). *Let X be an abelian variety of dimension g and α an automorphism of X of order $d \geq 3$ with $X_{\langle\alpha\rangle}$ finite and $\#\Phi_\alpha = \frac{\varphi(d)}{2}$. Then $\varphi(d)|2g$ and $(\mathbb{Q}(\xi), \Phi_\alpha)$ is a CM-field, there are $k = \frac{2g}{\varphi(d)}$ abelian varieties $X_1, \cdots, X_k$ of CM-type $(\mathbb{Q}(\xi_d), \Phi_\alpha)$ such that*

$$X \cong X_1 \times \cdots \times X_k,$$

and α decomposes into a product of automorphisms of the X_ν induced by primitive d-th roots of unities.

PROOF. According to the proof of Proposition 3.1 the automorphism α induces a $\mathbb{Q}(\xi_d)$-vector space structure on $H_1(X, \mathbb{Q})$. Moreover $H_1(X, \mathbb{Z})$ is a torsion free $\mathbb{Z}[\xi_d]$-module of rank k, since all eigenvalues of $\rho_r(\alpha)$ are primitive, $\mathrm{rk}_{\mathbb{Z}}\, \mathbb{Z}[\xi_d] = \varphi(d)$, and $\mathrm{rk}_{\mathbb{Z}} H_1(X, \mathbb{Z}) = 2g$. Now $\mathbb{Z}[\xi]$ is a Dedekind ring. Hence there are ideals $I_1, \cdots, I_k \subseteq \mathbb{Z}[\xi_d]$ such that

$$H_1(X, \mathbb{Z}) \cong I_1 \oplus \cdots \oplus I_k.$$

Denote $V_\nu := I_\nu \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(\xi_d)$. Then

$$H_1(X, \mathbb{Q}) = V_1 \oplus \cdots \oplus V_k$$

is a decomposition of $H_1(X, \mathbb{C})$ into one-dimensional $\mathbb{Q}(\xi_d)$-vector spaces, inducing an isomorphism of real tori:

$$X \cong X_1 \times \cdots \times X_k$$

with $X_\nu = V_\nu \otimes_{\mathbb{Q}} \mathbb{R}/I_\nu$. The analytic representation induces on the lattice the rational representation, implying $\rho_a(\alpha)I_\nu = \rho_r(\alpha)I_\nu \subseteq I_\nu$ and hence $\rho_a(\alpha)V_\nu \subseteq V_\nu$ for $\nu = 1, \cdots, k$. Hence by construction $\rho_a(\alpha)$ stabilizes the real tori X_ν.

According to Proposition 1.8 b) $(\mathbb{Q}(\xi_d), \Phi_\alpha)$ is a CM-field, since $\#\Phi_\alpha = \frac{\varphi(d)}{2} =: s$. The set Φ_α defines a complex structure J_ν on $V_\nu \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{Q}(\xi_d) \otimes_{\mathbb{Q}} \mathbb{R}$. It remains to show that the complex structure on $V_\nu \otimes_{\mathbb{Q}} \mathbb{R}$ induced by the complex structure of $H_1(X, \mathbb{R})$ coincides with J_ν.

Denoting $\Phi_\alpha = \{\lambda_1, \cdots, \lambda_s\}$, the complex structure J_ν on $V_\nu \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{Q}(\xi_\alpha) \otimes_{\mathbb{Q}} \mathbb{R}$ is defined by the following commutative diagram

$$\begin{array}{ccc} V_\nu \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{Q}(\xi_d) \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\phi_\nu} & \mathbb{C}^s \\ {\scriptstyle J_\nu}\downarrow & & \downarrow{\scriptstyle i\mathbf{1}_s} \\ V_\nu \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{Q}(\xi_d) \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\phi_\nu} & \mathbb{C}^s \end{array}$$

where ϕ_ν is defined by $(\lambda_1, \cdots, \lambda_s)$. Since $1, \xi_d, \cdots, \xi_d^{d-1}$ generate the $\mathbb{R}$-vector space $\mathbb{Q}(\xi) \otimes_{\mathbb{Q}} \mathbb{R}$, there are real numbers $r_0, \cdots, r_{d-1}$ such that the image of $1 \otimes 1 \in \mathbb{Q}(\xi_d) \otimes_{\mathbb{Q}} \mathbb{R} = V_\nu \otimes_{\mathbb{Q}} \mathbb{R}$ under J_ν is of the form

$$J_\nu(1 \otimes 1) = \sum_{j=0}^{d-1} \xi_d^j \otimes r_j$$

This implies

$$\begin{pmatrix} i \\ \vdots \\ i \end{pmatrix} = i\,\phi_\nu(1 \otimes 1) = \phi_\nu J_\nu(1 \otimes 1)$$

$$= \begin{pmatrix} \lambda_1 J_\nu(1\otimes 1) \\ \vdots \\ \lambda_s J_\nu(1\otimes 1) \end{pmatrix} = \begin{pmatrix} \lambda_1\left(\sum\limits_{j=1}^{d-1} \xi_d^j \otimes r_j\right) \\ \vdots \\ \lambda_s\left(\sum\limits_{j=1}^{d-1} \xi_d^j \otimes r_j\right) \end{pmatrix} = \sum_{j=1}^{d-1} r_j \begin{pmatrix} \lambda_1^j \\ \vdots \\ \lambda_s^j \end{pmatrix}$$

On the other hand $\lambda_1, \cdots, \lambda_s$ are exactly the eigenvalues of $\rho_a(\alpha)$. Hence this equation implies

$$i \cdot \mathrm{id}_{H_1(X,\mathbb{R})} = \sum_{j=0}^{d-1} r_j \rho_a(\alpha)^j$$

where $H_1(X, \mathbb{R})$ is considered as a complex vector space. This gives the assertion. □

As an immediate consequence we obtain

COROLLARY 3.3. *Suppose α is an automorphism of an abelian variety of order $d \geq 3$ with $\rho_a(\alpha) = \xi_d \cdot id$. Then $d = 3, 4$ or 6 and*

$$X = E \times \cdots \times E$$

where E denotes the elliptic curve admitting an automorphism of order d.

4. Abelian Threefolds with $\dim X_{\langle\alpha\rangle} = 0$

In this section we determine all pairs (X,α) where X is an abelian threefold and α an automorphism of X of order $d \geq 3$ such that the set $X_{\langle\alpha\rangle}$ is finite.

Suppose (X,α) is such a pair. According to Proposition 1.8 the number $\varphi(d)$ divides 6, hence

$$d = \operatorname{ord}\alpha \in \{3, 4, 6, 7, 9, 14, 18\}.$$

Moreover by loc. cit. the set of eigenvalues Φ_α of α consists of at least $\frac{\varphi(d)}{2}$ primitive non-complex conjugate d-th roots of unity. Replacing eventually α by a suitable power this gives:

Lemma 4.1. *The pair (d, Φ_α) is contained in the following list:*

d	Φ_α
3	$\{\rho\}$
	$\{\rho, \rho^2\}$
4	$\{i\}$
	$\{i, -i\}$
6	$\{-\rho\}$
	$\{-\rho, -\rho^2\}$

7	$\{\xi_7, \xi_7^2, \xi_7^3\} = \Phi_1^7$
	$\{\xi_7, \xi_7^2, \xi_7^4\} = \Phi_2^7$
14	$\{-\xi_7, -\xi_7^2, -\xi_7^3\}$
	$\{-\xi_7, -\xi_7^2, -\xi_7^4\}$
9	$\{\xi_9, \xi_9^2, \xi_9^4\} = \Phi_1^9$
	$\{\xi_9, \xi_9^4, \xi_9^7\} = \Phi_2^9$
18	$\{-\xi_9, -\xi_9^2, -\xi_9^4\}$
	$\{-\xi_9, -\xi_9^4, -\xi_9^7\}$

Here $\rho := \xi_3$, a primitive third root of unity.

If $\#\Phi_\alpha = \frac{\varphi(d)}{2}$ (that is $d \in \{3,4,6\}$ and $\#\Phi_\alpha = 1$ or $d \in \{7, 9, 14, 18\}$), there is a uniquely determined abelian threefold X admitting an automorphism α with eigenvalues Φ_α as in the lemma. To be more precise, according to the results of section 3 we have

Theorem 4.2. *Let (X,α) be an abelian threefold with an automorphism α of order d such that $X_{\langle\alpha\rangle}$ is finite and $\#\Phi_\alpha = \frac{\varphi(d)}{2}$. Then X admits a polarization H invariant with respect to α and replacing eventually α by a suitable power the data $(X, \Phi_\alpha,$ type of $H)$ are contained in the following list*

$d = ord\ \alpha$	X	Φ_α	*type of H*
3	$E_\rho \times E_\rho \times E_\rho$	$\{\rho\}$	$(3,3,3)$
4	$E_i \times E_i \times E_i$	$\{i\}$	$(4,4,4)$
6	$E_\rho \times E_\rho \times E_\rho$	$\{-\rho\}$	$(3,3,3)$
7	$X(\mathbb{Q}(\xi_7), \Phi_1^7)$	$\Phi_1^7 = \{\xi_7, \xi_7^2, \xi_7^3\}$	$(1,1,1)$
	$X(\mathbb{Q}(\xi_7), \Phi_2^7)$	$\Phi_2^7 = \{\xi_7, \xi_7^2, \xi_7^4\}$	$(1,1,7)$
14	$X(\mathbb{Q}(\xi_7), \Phi_1^7)$	$-\Phi_1^7$	$(1,1,1)$
	$X(\mathbb{Q}(\xi_7), \Phi_2^7)$	$-\Phi_2^7$	$(1,1,7)$
9	$X(\mathbb{Q}(\xi_9), \Phi_1^9)$	$\Phi_1^9 = \{\xi_9, \xi_9^2, \xi_9^4\}$	$(1,3,3)$
	$X(\mathbb{Q}(\xi_9), \Phi_2^9)$	$\Phi_2^9 = \{\xi_9, \xi_9^4, \xi_9^7\}$	$(1,1,1)$
18	$X(\mathbb{Q}(\xi_9), \Phi_1^9)$	$-\Phi_1^9$	$(1,3,3)$
	$X(Q(\xi_9), \Phi_9^2)$	$-\Phi_2^9$	$(1,1,1)$

Here E_τ denotes the elliptic curve $\mathbb{C}/\tau\mathbb{Z} + \mathbb{Z}$.

PROOF. This an immediate consequence of Theorem 3.2. For an explicit description of the of the abelian threefolds of CM-type (cases $d = 7, 14, 9$ or 18) in terms of period matrices see Section 3. □

We now turn to the remaining case, $\#\Phi_\alpha > \frac{\varphi(d)}{2}$: Suppose (X, H, α) is a polarized abelian threefold with an automorphism α of order $d \in \{3,4,6\}$, a finite set $X_{\langle\alpha\rangle}$ and $\#\Phi_\alpha = 2$. Then the automorphism α induces an embedding $\mathbb{Q}(\xi_d) \mapsto \mathrm{End}_{\mathbb{Q}}(X)$, i.e. X admits complex multiplication by $\mathbb{Q}(\xi_d)$. Such abelian threefolds were constructed by Shimura (see **[CAV]** Section 9.6). For the sake of completeness we sketch this construction:

Suppose $d = 3, 4$ or 6. Let $\mathcal{M} \subset \mathbb{Q}(\xi_d)^{\oplus 3} \subset \mathbb{C}^3$ be a free $\mathbb{Z}$-submodule of rank 6 and $T \in GL_3(\mathbb{Q}(\xi_d))$ a nondegenerate matrix satisfying

$$tr_{\mathbb{Q}(\xi_d)/\mathbb{Q}}\left({}^t\lambda T \mu\right) \in \mathbb{Z}$$

for all $\lambda, \mu \in \mathcal{M}$. Suppose moreover that T is skew hermitian of signature $(1,2)$, i.e. ${}^t\overline{T} = -T$ and

$$T = {}^tW \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \overline{W}$$

for some $W \in GL_3(\mathbb{C})$. (Below we give examples of such pairs $(\mathcal{M}, T)$).

Consider the complex manifold

$$\mathcal{H} := \left\{ \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1 \right\}$$

A point $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in \mathcal{H}$ defines an isomorphism of $\mathbb{R}$-vector spaces

$$J_z : \mathbb{C}^3 \to \mathbb{C}^3,\ v \mapsto \begin{pmatrix} (z, \mathbf{1}_2) & 0 \\ 0 & (1, {}^tz) \end{pmatrix} \begin{pmatrix} Wv \\ \overline{Wv} \end{pmatrix} = \begin{pmatrix} (z, \mathbf{1}_2)Wv \\ (1, {}^tz)\overline{Wv} \end{pmatrix}$$

as well as an hermitian form $H_z : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$ by the matrix

$$2\begin{pmatrix} (\mathbf{1} - \overline{z}{}^tz)^{-1} & 0 \\ 0 & (1 - {}^t\overline{z}z)^{-1} \end{pmatrix} = 2\begin{pmatrix} 1-|z_1|^2 & \overline{z}_1 z_2 & 0 \\ z_1\overline{z}_2 & 1-|z_2|^2 & 0 \\ 0 & 0 & 1-|z_1|^2-|z_2|^2 \end{pmatrix}^{-1}$$

PROPOSITION 4.3. *$(X_z = \mathbb{C}^3/J_z(\mathcal{M}), H_z)$ is a polarized abelian threefold admitting an embedding $\iota : \mathbb{Q}(\xi_d) \mapsto \mathrm{End}_{\mathbb{Q}}(X_z)$ such that*

i) *$\rho_a\iota(a)$ is equivalent to $\left(\begin{smallmatrix} a\mathbf{1}_2 & 0 \\ 0 & \overline{a}\end{smallmatrix}\right)$ for all $a \in \mathbb{Q}(\xi_d)$;*
ii) *the Rosati involution with respect to H_z restricts to complex conjugation on $\mathbb{Q}(\xi_d)$.*

In particular X_z admits an automorphism α of order $d \in \{3,4,6\}$ such that $(X_z)_{<\alpha>}$ is finite, $\#\Phi_\alpha = 2$ and H_z is invariant with respect to α.

In more sophisticated terms this means that the polarized abelian variety (X_z, H_z) admits multiplication by $\mathbb{Q}(\xi_d)$ of signature $(1,2)$.

PROOF. Obviously X_z is a complex torus, $\mathcal{M}$ being a lattice in $\mathbb{C}^3$ and J_z being an isomorphism of $\mathbb{R}$-vector spaces. Moreover by definition of $\mathcal{H}$, H_z is a positive definite hermitian form. In order to show that H_z defines a polarization on X_z it remains to show that its imaginary part Im H_z takes only integer values on the lattice $J_z(\mathcal{M})$. To see this we need the following identity:

$$\begin{pmatrix} {}^tz \\ \mathbf{1}_2 \end{pmatrix} (\mathbf{1}_2 - \overline{z}\,{}^tz)^{-1}(\overline{z}, \mathbf{1}_2) - \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} \left(1 - {}^tz\overline{z}\right)^{-1}(1, {}^tz) = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \qquad (*)$$

For the proof of (*) we first observe that $\overline{z}(1 - {}^tz\overline{z})^{-1} = (\mathbf{1}_2 - \overline{z}{}^tz)^{-1}\overline{z}$ and equivalently $(1 - {}^tz\,\overline{z})^{-1}\,{}^tz = {}^tz\,(\mathbf{1}_2 - \overline{z}{}^tz)^{-1}$. Using this we have:

$$\text{left hand side of } (*) = \begin{pmatrix} {}^tz(\mathbf{1}_2-\overline{z}{}^tz)^{-1}\overline{z} & {}^tz(\mathbf{1}_2-\overline{z}{}^tz)^{-1} \\ (\mathbf{1}_2-\overline{z}{}^tz)^{-1}\overline{z} & (\mathbf{1}_2-\overline{z}{}^tz)^{-1} \end{pmatrix} - \begin{pmatrix} (1-{}^tz\,\overline{z})^{-1} & (1-{}^tz\,\overline{z})^{-1}\,{}^tz \\ \overline{z}(1-{}^tz\overline{z})^{-1} & \overline{z}(1-{}^tz\overline{z})^{-1}\,{}^tz \end{pmatrix}$$

$$= \begin{pmatrix} (1-{}^tz\overline{z})^{-1} & 0 \\ 0 & (\mathbf{1}_2-\overline{z}{}^tz)^{-1} \end{pmatrix} \left[\begin{pmatrix} {}^tz\overline{z} & {}^tz \\ \overline{z} & \mathbf{1}_2 \end{pmatrix} - \begin{pmatrix} 1 & {}^tz \\ \overline{z} & \overline{z}{}^tz \end{pmatrix}\right]$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} = \text{ right hand side of } (*)\ .$$

Now we have for all $\lambda, \mu \in \mathcal{M}$:

$$\begin{aligned} \text{Im}\quad & H_z(J_z(\lambda), J_z(\mu)) = \\ &= 2\ \text{Im}\ {}^t\begin{pmatrix} W\lambda \\ \overline{W\lambda} \end{pmatrix} \begin{pmatrix} \binom{{}^tz}{\mathbf{1}_2}(\mathbf{1}_2-\overline{z}{}^tz)^{-1}(\overline{z},\mathbf{1}_2) & 0 \\ 0 & \binom{1}{z}(1-{}^t\overline{z}z)(1,{}^t\overline{z}) \end{pmatrix} \begin{pmatrix} \overline{W\mu} \\ W\mu \end{pmatrix} \\ &= 2\ \text{Im}\ \left[{}^t\lambda{}^tW\left[\binom{{}^tz}{\mathbf{1}_2}(\mathbf{1}_2 - \overline{z}{}^tz)^{-1}(\overline{z}, \mathbf{1}_2) - \binom{1}{\overline{z}}(1 - {}^tz\overline{z})(1, {}^tz)\right]\overline{W\overline{\mu}}\right] \\ &\overset{(*)}{=} 2\ \text{Im}\ \left[{}^t\lambda{}^tW\begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\overline{W\overline{\mu}}\right] \\ &= 2\ \text{Im}\ [i{}^t\lambda T\overline{\mu}] \\ &= 2\text{Re}\ [{}^t\lambda T\overline{\mu}] \\ &= tr_{\mathbb{Q}(\xi_d)/\mathbb{Q}}({}^t\lambda T\overline{\mu}) \in \mathbb{Z} \end{aligned}$$

This proves the first statement of the Proposition. As for the second statement consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{Q}(\xi_d)^{\oplus 3} & \xrightarrow{J_z} & \mathbb{C}^3 \\ {\scriptstyle a\mathbf{1}_3}\downarrow & & \downarrow \left(\begin{smallmatrix} a\mathbf{1}_2 & 0 \\ 0 & \overline{a}\end{smallmatrix}\right) \\ \mathbb{Q}(\xi_d)^{\oplus 3} & \xrightarrow{J_z} & \mathbb{C}^3 \end{array}$$

where $a \in \mathbb{Q}(\xi_d)$. This shows that scalar multiplication by $\mathbb{Q}(\xi_d)$ induces an embedding $\iota : \mathbb{Q}(\xi_d) \mapsto \text{End}_{\mathbb{Q}}(X_z)$ satisfying property i).

Finially using the block matrix structure of H_z and the above diagram one immediately sees that for all $a \in \mathbb{Q}(\xi_d)$ and $v, w \in \mathbb{C}^3$:

$$H_z(\iota(a)v, w) = H_z(v, \iota(\overline{a})w).$$

Hence the Rosati-involution, which is the adjoint operator on $\operatorname{End}_{\mathbb{Q}}(X)$ with respect to H_z, restricts to complex conjugation on $\mathbb{Q}(\xi_d)$. This completes the proof. □

Note that the choice of the pair $(\mathcal{M}, T)$ determines the type of the polarization H_y. It is easy to see that every type (d_1, d_2, d_3) occurs at least once.

According to [**CAV**], Proposition 9.6.5 any polarized abelian threefold with multiplication by $\mathbb{Q}(\xi_d)$ of signature $(1,2)$ is isomorphic to (X_z, H_z) for some pair $(\mathcal{M}, T)$ as above and some $z \in \mathcal{H}$. Hence we conclude

THEOREM 4.4. *Let (X, H, α) be a polarized abelian threefold with an automorphism α of order $d = 3, 4$ or 6 such that $X_{\langle\alpha\rangle}$ is finite, $\#\Phi_\alpha = 2$ and H is invariant with respect to α. Then there is a $\mathbb{Z}[\xi_d]$-module $\mathcal{M} \subset \mathbb{Q}(\xi_d)^{\oplus 3}$ of rank 6 and a nondegenerate skew-hermitian matrix $T \in GL_3(\mathbb{Q}(\xi_d))$ of signature $(1,2)$ satisfying $tr_{\mathbb{Q}(\xi_d)/\mathbb{Q}}({}^t a T \overline{b}) \in \mathbb{Z}$ for all $a, b \in \mathcal{M}$, such that (with the notation of Proposition 4.3)*

$$(X, H, \alpha) \sim (X_z = \mathbb{C}^3 / J_z(\mathcal{M}),\ H_z,\ \iota(\xi_d)).$$

PROOF. By what we have said above (X, H) is isomorphic to (X_z, H_z) for some pair $(\mathcal{M}, T)$ as above and some $z \in \mathcal{H}$. According to Proposition 4.3 there is an embedding $\iota : \mathbb{Q}(\xi_d) \mapsto \operatorname{End}_{\mathbb{Q}}(X_z) = \operatorname{End}_{\mathbb{Q}}(X)$ such that $\iota(\xi_d)$ admits exactly two complex conjugate eigenvalues. We have to show that $\alpha = \iota(\xi_d) \in \operatorname{End}(X)$. But thus is equivalent to $\mathcal{M}$ being a $\mathbb{Z}[\xi_d]$-module. Finally using again the block-matrix structure of H_z it is easy to see that $\alpha^* H_z = H_z$. □

EXAMPLES 4.5. a) *Case $d = 3$ or 6:*

Choose $\mathcal{M} = \mathbb{Z}[\xi_d]^{\oplus 3} = \mathbb{Z}[\rho]^{\oplus 3} \subset \mathbb{Q}(\rho)$, $T = \begin{pmatrix} (\rho - \rho^2) & 0 \\ 0 & (\rho^2 - \rho)\mathbf{1}_2 \end{pmatrix}$, and $W = \sqrt[4]{3}\,\mathbf{1}_3$. Then

$${}^t W \begin{pmatrix} i & 0 \\ 0 & -i\mathbf{1}_2 \end{pmatrix} \overline{W} = \begin{pmatrix} \sqrt{3}i & 0 \\ 0 & -\sqrt{3}i\mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} \rho - \rho^2 & 0 \\ 0 & (\rho^2 - \rho)\mathbf{1}_2 \end{pmatrix} = T.$$

Suppose $z = \binom{z_1}{z_2} \in \mathcal{H}$. An immediate computation shows that $J_z(\mathcal{M}) = \pi_3 \mathbb{Z}^6$ with

$$\pi_3 = \sqrt[4]{3} \begin{pmatrix} z_1 & 1 & 0 & \rho z_1 & \rho & 0 \\ z_2 & 0 & 1 & \rho z_2 & 0 & \rho \\ 1 & z_1 & z_2 & \rho^2 & \rho^2 z_1 & \rho^2 z_2 \end{pmatrix}$$

Hence $(X = \mathbb{C}^3 / \pi_3 \mathbb{Z}^6, H_z)$ is a polarized abelian threefold with automorphism $\alpha = \begin{pmatrix} \rho\mathbf{1}_2 & 0 \\ 0 & \rho^2 \end{pmatrix}$ of order 3 and an automorphism $-\alpha$ of order 6.

b) *Case $d = 4$:* Choose $\mathcal{M} = \mathbb{Z}[i]^{\oplus 3} \in \mathbb{Q}(i)$, and $T = \begin{pmatrix} i & 0 \\ 0 & -i\mathbf{1}_2 \end{pmatrix}$. For $z = \binom{z_1}{z_2} \in \mathcal{H}$ we have $J_z(\mathcal{M}) = \pi_4 \mathbb{Z}^6$ with

$$\pi_4 = \begin{pmatrix} z_1 & 1 & 0 & iz_1 & i & 0 \\ z_2 & 0 & 1 & iz_2 & 0 & i \\ 1 & z_1 & z_2 & -i & -iz_1 & -iz_2 \end{pmatrix}.$$

Hence $(X = \mathbb{C}^3 / \pi_4 \mathbb{Z}^6, H_z)$ is a polarized abelian threefold with automorphism $\alpha = \begin{pmatrix} i\mathbf{1}_2 & 0 \\ 0 & -i \end{pmatrix}$ of order 4.

5. Abelian Threefolds X with $\dim X_{\langle\alpha\rangle} > 0$

In this section we study abelian threefolds X admitting an automorphism α of order $d \geq 3$ with $\dim X_G \geq 1$, where G denotes the cyclic group $\langle\alpha\rangle$ generated α. According to the Decomposition Theorem we have to distinguish 2 cases:

Case I: Exactly 2 of the 3 eigenvalues of $\rho_a(\alpha)$ have the same order. Let the orders of the eigenvalues be $d_1 \neq d_2$, where d_2 occurs twice. In this case there are G-stable abelian subvarieties E (of dimension 1) and A (of dimension 2) of X such that the restriction $\alpha_E := \alpha|E$ is of order d_1 and $\alpha_A := \alpha|A$ is of order d_2 and such that the addition map $\mu : E \times A \to X$ is an isogeny.

Case II: The orders d_1, d_2 and d_3 of the 3 eigenvalues of $\rho_a(\alpha)$ are pairwise different. In this case there are G-stable abelian subvarieties E_1, E_2, and E_3 (of dimension 1) in X such that $\alpha_{E_i} := \alpha|E_i$ is of order d_i for $i = 1, 2, 3$ and such that the addition map $\mu : E_1 \times E_2 \times E_3 \to X$ is an isogeny.

Case I

Let the notation be as in Case I above. Moreover let e denote the greatest common divisor of d_1 and d_2. The kernel of the isogeny

$$\mu : E \times A \to X$$

has the following properties:

PROPOSITION 5.1. *(i)* $\mathrm{Ker}(\mu) \subseteq \mathrm{Fix}\,\alpha_E^e \times \mathrm{Fix}\,\alpha_A^e$
(ii) The projections $p : \mathrm{Ker}(\mu) \to E$ *and* $q : \mathrm{Ker}(\mu) \to A$ *are injective. Moreover* $\mathrm{Ker}(\mu)$ *is* $(\alpha_E \times \alpha_A)$*-stable.*

PROOF. For $(x, y) \in \mathrm{Ker}(\mu)$ we have $y = -x$ (note that we consider E and A as subvarieties of X). Hence the projections p and q are injective. Moreover $0 = \mu((\alpha_E \times \alpha_A)^{d_1}(x, -x)) = x - \alpha^{d_1}(x)$ and $0 = \mu(\alpha_E \times \alpha_A)^{d_2}(x, -x)) = \alpha^{d_2}(x) - x$ imply

$$x \in \mathrm{Fix}\,(\alpha^{d_1}) \cap \mathrm{Fix}\,(\alpha^{d_2}) \cap E = \mathrm{Fix}\,(\alpha^e) \cap E = \mathrm{Fix}\,\alpha_E^e.$$

Similarly $y \in \mathrm{Fix}\,\alpha_A^e$. The last assertion is trivial. □

PROPOSITION 5.2. *Let the notation be as above. Suppose*

(i) $\frac{d_1}{e}$ *is not a prime power or*
(ii) $A_{\langle\alpha_A\rangle}$ *is finite and* $\frac{d_2}{e}$ *is not a prime power or*
(iii) $E_{\langle\alpha_E^e\rangle}$ *and* $A_{\langle\alpha_A^e\rangle}$ *are finite and* $d_1 \neq e \neq d_2$,

then $\mu : E \times A \to X$ *is an isomorphism.*

PROOF. Suppose $\frac{d_1}{e}$ is not a prime power. Then α_E^e is of order $\frac{d_1}{e} \neq 1$ and hence $E_{\langle\alpha_E^e\rangle}$ is finite. Thus Corollary 1.7 implies $\#\mathrm{Fix}\,\alpha_E^e = 1$. Now Proposition 5.1 implies assertion (i). The same proof works for assertion (ii). Finally in case (iii)

$$\mathrm{Ker}(\mu) \simeq p(\mathrm{Ker}(\mu)) \subseteq \mathrm{Fix}\,(\alpha_E^e) \subseteq E_{\frac{d_1}{e}} \simeq \left(\mathbb{Z}/\tfrac{d_1}{e}\mathbb{Z}\right)^2$$

and

$$\mathrm{Ker}(\mu) \simeq q(\mathrm{Ker}(\mu)) \subseteq \mathrm{Fix}\,(\alpha_A^e) \subseteq A_{\frac{d_2}{e}} \simeq \left(\mathbb{Z}/\tfrac{d_2}{e}\mathbb{Z}\right)^4$$

Since $\gcd\left(\frac{d_1}{e}, \frac{d_2}{e}\right) = 1$, this implies $\#\mathrm{Ker}(\mu) = 1$. Hence μ is an isomorphism. □

In order to construct abelian threefolds X with an automorphism α with $\dim X_{\langle\alpha\rangle} \geq 1$ we use the following proposition which may be considered as a sort of converse statement to the Decomposition Theorem in this case.

PROPOSITION 5.3. *Let E be an elliptic curve with an automorphism α_E of order d_1 and A an abelian surface with an automorphism α_A of order d_2 and assume $d_1 \neq d_2$. Denote $e = \gcd(d_1, d_2)$ and let K be a finite subgroup of $E \times A$ with*

$$K \subseteq \mathrm{Fix}\,(\alpha_E^e) \times \mathrm{Fix}\,(\alpha_A^e) \ \ \textit{and} \ \ (\alpha_E \times \alpha_A)K \subseteq K,$$

such that the projections $p : K \to E$ and $q : K \to A$ are injective. Then the abelian variety $X = E \times A/K$ admits a unique automorphism α such that E and A, considered as abelian subvarieties of X, are α-stable with $\alpha|E = \alpha_E$ and $\alpha|A = \alpha_A$. The isogeny $E \times A \to X = E \times A/K$ is given by the addition map. Moreover $\mathrm{ord}(\alpha) = \mathrm{lcm}(d_1, d_2)$ *and* $\dim X_{\langle\alpha\rangle} \geq 1$.

The proof is immediate, so we omit it. □

According to Example 1.9 the degrees of the nontrivial automorphisms of an elliptic curve are 2, 3, 4 and 6. According to Example 1.10 the degrees of the the nontrivial automorphisms α of an abelian surface X with $X_{\langle\alpha\rangle}$ finite are $2, 3, 4, 5, 6, 8, 10$ and 12. Hence the possible degrees d_1 and d_2 are $d_1 \in \{1, 2, 3, 4, 6\}$ and $d_2 \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. Moreover we may assume $d_E \neq d_A$ and $\mathrm{lcm}(d_1, d_2) \geq 3$, since we are interested in automorphisms of degree $d \geq 3$. Hence if $E, \alpha_{E_1}, A, \alpha_A$ and K are given and X and α are constructed as in Proposition 5.3, we have for the degree d of α

$$d \in \{3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30\}.$$

We may apply Proposition 5.2 to deduce that in some cases the subgroup K in Proposition 5.3 is necessarily trivial, implying $X = E \times A$ and $\alpha = \alpha_E \times \alpha_A$.

This is the case for:

$$\begin{aligned}(d_1, d_2) \ \in \ \{&(1,6), (1,10), (1,12), (2,3), (2,5), (2,12), (3,2), (3,4),\\ &(3,5), (3,8), (3,10), (4,3), (4,5), (4,6), (4,10), (6,1),\\ &(6,4), (6,5), (6,8), (6,10), (6,12)\}\end{aligned}$$

It remains to consider the cases of the following table. We will see below that apart from the trivial case $K = \{(0,0)\}$ there are also nontrivial subgroups $K \subset E \times A$ such that $\alpha_E \times \alpha_A$ induces an automorphism α on $X = E \times A/K$.

As for the notation in Table 5.4 below: E denotes an arbitrary elliptic curve (so $E = \mathbb{C}/\tau\mathbb{Z} + \mathbb{Z}$ for some $\tau \in \mathfrak{H}_1$), A an arbitrary abelian surface (so $A = \mathbb{C}^2/(Z, \delta)\mathbb{Z}^4$ with $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right) \in \mathfrak{H}_2$ and some type of polarization $\delta = \left(\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right)$), E_j denotes the elliptic curve with j-invariant j, and $X(\mathbb{Q}(\xi_d))$ one of the abelian surfaces $X(\mathbb{Q}(\xi_d), (\xi_d^{a_1}, \xi_d^{a_2})) = \mathbb{Q}(\xi_d) \otimes_{\mathbb{Q}} \mathbb{R}/\mathcal{O}$ of CM-type $(\xi_d^{a_1}, \xi_d^{a_2})$ for $d = 5, 8, 12$. It is well known (see e.g. [**FU**]) that the elliptic curves and abelian surfaces in the 3rd and 4th column of Table 5.4 below are the only abelian varieties X of dimension 1 and 2 admitting an automorphism α_E and α_A of the indicated order with $E_{\langle\alpha_E\rangle}$ and $A_{\langle\alpha_A\rangle}$ finite. By a slight abuse of notation we denote by ξ_d the automorphism on E and A induced by multiplication by a dth primitive root of unity. Finally recall that $e = \gcd(d_1, d_2)$ and $d = \mathrm{ord}(\alpha) = \mathrm{lcm}(d_1, d_2)$ is the order of the automorphism α on $X = E \times A/K$.

TABLE 5.4.

No	(d_1, d_2)	E	A	α_E	α_A	e	d
1	$(1,3)$	E	$E_0 \times E_0$	id	ρ	1	3
2	$(1,4)$	E	$E_{1728} \times E_{1728}$	id	i	1	4
3	$(1,5)$	E	$X(\mathbb{Q}(\xi_5))$	id	ξ_5	1	5
4	$(1,8)$	E	$X(\mathbb{Q}(\xi_8))$	id	ξ_8	1	8
5	$(2,4)$	E	$E_{1728} \times E_{1728}$	-id	i	2	4
6	$(2,6)$	E	$E_0 \times E_0$	$-$id	$-\rho$	2	6
7	$(2,8)$	E	$X(\mathbb{Q}(\xi_8))$	$-$id	ξ_8	2	8
8	$(2,10)$	E	$X(\mathbb{Q}(\xi_5))$	$-$id	$-\xi_5$	2	10
9	$(3,1)$	E_0	A	ρ	id	1	3
10	$(3,6)$	E_0	$E_0 \times E_0$	ρ	$-\rho$	3	6
11	$(3,12)$	E_0	$X(\mathbb{Q}(\xi_{12}))$	ρ	ξ_{12}	3	12
12	$(4,1)$	E_{1728}	A	i	id	1	4
13	$(4,2)$	E_{1728}	A	i	$-$id	2	4
14	$(4,8)$	E_{1728}	$X(\mathbb{Q}(\xi_8))$	i	ξ_8	4	8
15	$(4,12)$	E_{1728}	$X(\mathbb{Q}(\xi_{12}))$	i	ξ_{12}	4	12
16	$(6,2)$	E_0	A	$-\rho$	$-$id	2	6
17	$(6,3)$	E_0	$E_0 \times E_0$	$-\rho$	ρ	3	6
18	$(6,12)$	E_0	$X(\mathbb{Q}(\xi_{12}))$	$-\rho$	ξ_{12}	6	12

Below we give examples of an abelian threefolds $X = \mathbb{C}^3/(Z,D)\mathbb{Z}^6$ with an automorphism α for every type Nos 1 to 18 of Table 5.4. More examples are given in the thesis [**SCH**]. Moreover D. Schmidt shows that in several of the above cases every pair (X, α) is isomorphic to one of the examples given below. In the examples we describe the threefold X in terms of the period matrix, normalized in the form $\pi = (Z, D)$ with an element Z of the Siegel upper half space $\mathfrak{H}_3$ and an integer valued diagonal matrix D indicating the type of the polarization of X. The automorphism α of X is given in terms of its analytic representation $\rho_a(\alpha)$ and its rational representation $\rho_r(\alpha)$ with respect to the coordinates given by the period matrix. Hence the proof that α is an automorphism of X consists just in checking the equation

$$\rho_a(\alpha)(Z,D) = (Z,D)\rho_r(\alpha). \qquad (*)$$

Moreover we indicate the isomorphism type of the subgroup K in each case. With this notation the following list gives examples of abelian threefolds with automorphism (X, α) corresponding to Nos 1 to 18 of Table 5.4:

EXAMPLES 5.5.

No 1: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau-1+\rho}{3} & 0 & \rho-1 & 1 & 0 & 0 \\ 0 & 1+\rho & 0 & 0 & 1 & 0 \\ \rho-1 & 0 & -3\rho^2 & 0 & 0 & 3 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & 0 & \frac{-1+\rho}{3} \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/3\mathbb{Z}$.

No 2: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau-1+i}{2} & 0 & -1+i & 1 & 0 & 0 \\ 0 & i & 0 & 0 & 1 & 0 \\ -1+i & 0 & 2i & 0 & 0 & 2 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & 0 & \frac{-1+i}{2} \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No 3: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau+16\xi_5+4\xi_5^2+12\xi_5^3}{5} & -6-2\xi_5^2-2\xi_5^3 & 5+4\xi_5+2\xi_5^2+4\xi_5^3 & 1 & 0 & 0 \\ -6-2\xi_5^2-2\xi_5^3 & 5\xi_5^2-5\xi_5^3 & -5-5\xi_5^2 & 0 & 5 & 0 \\ 5+4\xi_5+2\xi_5^2+4\xi_5^3 & -5-5\xi_5^2 & 5+5\xi_5+5\xi_5^2+5\xi_5^3 & 0 & 0 & 5 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & \frac{-6-2\xi_5^2-2\xi_5^3}{5} & \frac{-6+4\xi_5+2\xi_5^3}{5} \\ 0 & \xi_5^2-\xi_5^3 & -1-\xi_5^3 \\ 0 & -1-\xi_5^2 & \xi_5+\xi_5^3 \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 & -5 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/5\mathbb{Z}$

No 4: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau}{2}+1+i & -1-i & -1-i & 1 & 0 & 0 \\ -1-i & 2i & 0 & 0 & 2 & 0 \\ -1-i & 0 & 2i & 0 & 0 & 2 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & 0 & \frac{1-i}{2} \\ 0 & 0 & i \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \rho_r(\alpha) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No 5: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau}{2}-1+i & -1+i & -1+i & 1 & 0 & 0 \\ -1+i & 2i & 0 & 0 & 2 & 0 \\ -1+i & 0 & 2i & 0 & 0 & 2 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} -1 & \frac{1+i}{2} & \frac{1+i}{2} \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & -1 & 2 & 2 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No 6: Let (X, α) be as in No 1. Then $(X, -\alpha)$ is an example of for No 6 of Table 5.4.

No 7: Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau}{2}+1+i & -1-i & -1-i & 1 & 0 & 0 \\ -1-i & 2i & 0 & 0 & 2 & 0 \\ -1-i & 0 & 2i & 0 & 0 & 2 \end{pmatrix}$ with $\tau \in \mathfrak{H}_1$.
Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} -1 & -1 & -\frac{1+i}{2} \\ 0 & 0 & i \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 2 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No 8: Let (X, α) be as in No 3. Then $(X, -\alpha)$ is an example for No 8 of Table 5.4.

No 9: Periodmatrix: $\pi = \begin{pmatrix} \frac{-1+\rho}{3} & 1 & 0 & 1 & 0 & 0 \\ 1 & 3z_1 & -3z_2 & 0 & 3 & 0 \\ 0 & -3z_2 & 3z_3 & 0 & 0 & 3 \end{pmatrix}$ with $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_r(\alpha) \begin{pmatrix} -2 & 3 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/3\mathbb{Z}$

No 10: Periodmatrix: $\pi = \begin{pmatrix} \frac{1}{2}+\rho & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & \rho & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2}+\rho & 0 & 0 & 1 \end{pmatrix}$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} -\frac{1}{2}-\rho & 0 & \frac{1}{2} \\ 0 & -\rho & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2}-\rho \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

No 11: Periodmatrix: $\pi = \begin{pmatrix} \frac{-5-7\rho^2}{3} & -\frac{20+16\rho^2}{3} & 8\,\frac{2+\rho^2}{3} & 1 & 0 & 0 \\ -\frac{20+16\rho^2}{3} & -\frac{56+40\rho^2}{3} & 20\,\frac{1-\rho}{3} & 0 & 4 & 0 \\ 8\,\frac{2+\rho^2}{3} & 20\,\frac{1-\rho}{3} & -16\,\frac{2+\rho^2}{3} & 0 & 0 & 4 \end{pmatrix}$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} -\frac{1+5\rho^2}{3} & \rho^2 & \frac{1+2\rho^2}{3} \\ -4\,\frac{1+2\rho^2}{3} & 2\rho^2 & \frac{1+5\rho^2}{3} \\ 4\,\frac{2+\rho^2}{3} & \rho & -2\,\frac{1+2\rho^2}{3} \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 3 & 0 & 4 & 3 & -4 & 0 \\ -1 & 0 & -2 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 \\ -5 & -4 & 0 & -2 & 0 & -4 \\ -2 & -3 & 2 & 0 & -2 & -3 \\ 2 & 4 & -3 & 0 & 1 & 2 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

No 12: Periodmatrix: $\pi = \begin{pmatrix} \frac{i-1}{2} & 1 & 0 & 1 & 0 & 0 \\ 1 & 2z_1 & -2z_2 & 0 & 2 & 0 \\ 0 & -2z_2 & 2z_3 & 0 & 0 & 2 \end{pmatrix}$ with $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} -1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No 13: Let (X, α) be as in No 12. Then $(X, -\alpha^3)$ is an example for No 13 of Table 5.4.

No 14: Periodmatrix: $\pi = \begin{pmatrix} \frac{-1+3i}{5} & \frac{4-2i}{5} & \frac{4-2i}{5} & 1 & 0 & 0 \\ \frac{4-2i}{5} & \frac{-6+8i}{5} & \frac{-6-2i}{5} & 0 & 2 & 0 \\ \frac{4-2i}{5} & \frac{-6-2i}{5} & \frac{-6+8i}{5} & 0 & 0 & 2 \end{pmatrix}$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} \frac{1+2i}{5} & \frac{-1-2i}{5} & 0 \\ \frac{-4+2i}{5} & \frac{-1+3i}{5} & i \\ \frac{6+2i}{5} & \frac{4+3i}{5} & 0 \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 2 \\ 1 & -1 & -1 & 1 & 1 & 2 \\ 1 & 0 & -1 & 1 & 1 & 1 \\ -2 & 2 & 2 & -1 & -2 & -2 \\ 1 & -1 & -2 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 2 & 1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

No 15: Periodmatrix: $\pi = \begin{pmatrix} \frac{i}{10} & \frac{9}{10} & -\frac{18}{5} & 1 & 0 & 0 \\ \frac{9}{10} & 99\frac{i}{10} & 27\frac{i}{5} & 0 & 9 & 0 \\ -\frac{18}{5} & 27\frac{i}{5} & 27\frac{i}{5} & 0 & 0 & 27 \end{pmatrix}$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} 7\frac{i}{10} & \frac{1}{10} & -\frac{2}{15} \\ \frac{63}{10} & 11\frac{i}{10} & -4\frac{i}{5} \\ \frac{9}{5} & 3\frac{i}{5} & \frac{i}{5} \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 0 & 9 & -27 & 7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -6 \\ -1 & 0 & 0 & 0 & 0 & 7 \\ -4 & 0 & 0 & 0 & 0 & 27 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 \end{pmatrix}$$

$K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

No 16: Let (X, α) be as in No 9. Then $(X, -\alpha^2)$ is an example for No 16 of Table 5.4.

No 17: Let (X, α) be as in No 10. Then $(X, -\alpha)$ is an example for No 17 of Table 5.4.

No 18: Periodmatrix: $\pi = \begin{pmatrix} -\frac{5+7\rho^2}{3} & -\frac{14+16\rho^2}{3} & \frac{10+8\rho^2}{3} & 1 & 0 & 0 \\ -\frac{14+16\rho^2}{3} & -\frac{44+40\rho^2}{3} & \frac{28+20\rho^2}{3} & 0 & 4 & 0 \\ \frac{10+8\rho^2}{3} & \frac{28+20\rho^2}{3} & -\frac{20+16\rho^2}{3} & 0 & 0 & 4 \end{pmatrix}$.

Analytic and rational representation of α:

$$\rho_a(\alpha) = \begin{pmatrix} \frac{2-5\rho^2}{3} & -\frac{1}{2}+\rho^2 & \frac{-1+4\rho^2}{6} \\ \frac{2-8\rho^2}{3} & -1+2\rho^2 & \frac{-2+5\rho^2}{3} \\ \frac{2+4\rho^2}{3} & -\rho^2 & \frac{1-4\rho^2}{3} \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 2 & 4 & 0 & 3 & -4 & 0 \\ -1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & -4 & -4 \\ -1 & -1 & 0 & 0 & -2 & -3 \\ 1 & 2 & -1 & 0 & 1 & 2 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z} \times Z/2\mathbb{Z}$

Case II:

Let α be an automorphism of order $d \geq 3$ of an abelian threefold X with $\dim X_{\langle\alpha\rangle} \geq 1$ such that the eigenvalues of $\rho_a(\alpha)$ have orders $d_1 < d_2 < d_3$. Then there are α-stable abelian subvarieties E_1, E_2 and ρ of dimension 1 in X, such that $\alpha_{E_i} := \alpha|E_i$ is of order d_i for $i = 1, 2, 3$ and the addtion map

$$\mu : E_1 \times E_2 \times E_3 \to X$$

is an isogeny. Let K denote the kernel of the isogeny μ. According to Example 1.9 we have $\operatorname{ord}(\alpha_{E_i}) \in \{1, 2, 3, 4, 6\}$. Hence we have to study the following cases (as always d denotes the order of α, and let E and E' denote arbitrary elliptic curves):

TABLE 5.6.

No	(d_1, d_2, d_3)	E_1	E_2	E_3	d
a)	(1, 2, 3)	E	E'	E_0	6
b)	(1, 2, 4)	E	E'	E_{1728}	4
c)	(1, 2, 6)	E	E'	E_0	6
d)	(1, 3, 4)	E	E_0	E_{1728}	12
e)	(1, 3, 6)	E	E_0	E_0	6
f)	(1, 4, 6)	E	E_{1728}	E_0	12
g)	(2, 3, 4)	E	E_0	E_{1728}	12
h)	(2, 3, 6)	E	E_0	E_0	6
i)	(2, 4, 6)	E	E_{1728}	E_0	12
12)	(3, 4, 6)	E_0	E_{1728}	E_0	12

The construction of pairs (X, α) as in Table 5.6 is very similar to the construction of Examples 5.5 in Case I. In fact one uses a result analogous to Proposition 5.3 above. We omit the details here (see [**SCH**]). Of course there are always examples with trivial K, that is an isomorphism $\mu : E_1 \times E_2 \times E_3 \to X$. According to Proposition 1.4 we have $\dim X_{\langle \alpha \rangle} \geq 1$ in these cases. Below we give an example for each case No a) to k) with K nontrivial. As above we give a period matrix in normalized form, the corresponding analytic and rational representations and the isomorphism type of K. Let τ and τ' be elements of the upper half plane $\mathfrak{H}_1$ such that $E \sim \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $E' \sim \mathbb{C}/\mathbb{Z} + \tau'\mathbb{Z}$.

EXAMPLES 5.7.

No a): Periodmatrix: $\pi = \begin{pmatrix} \frac{2\tau-1+\rho}{3} & \tau & -2\tau & 1 & 0 & 0 \\ \tau & \frac{\tau'+3\tau}{2} & -3\tau & 0 & 1 & 0 \\ -2\tau & -3\tau & 6(\tau+1) & 0 & 0 & 3 \end{pmatrix}$

Analytic and rational representation of the automorphism α:

$$\rho_a(\alpha) \begin{pmatrix} \rho & 0 & \frac{\rho-1}{3} \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} -2 & 0 & 6 & 3 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & -1 & -3 \\ 2 & 2 & -4 & -2 & 0 & -1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/6\mathbb{Z}$

No b): Periodmatrix: $\pi = \begin{pmatrix} 2\tau' & -2\tau' & 0 & 1 & 0 & 0 \\ -2\tau' & 2\tau'+2\tau & -2\tau & 0 & 1 & 0 \\ 0 & -2\tau & 1+2\tau+i & 0 & 0 & 2 \end{pmatrix}$.

Analytic and rational representation of the automorphism α:

$$\rho_a(\alpha) = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ -1+i & -1+i & i \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} -1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}$$

$K \simeq (\mathbb{Z}/2\mathbb{Z})^2$

No c): Let (X, α) be the pair of No a). Then $(X, -\alpha)$ is an example for No c) of Table 5.6.

No d): Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau-1+\rho}{3} & 0 & -1+\rho & 1 & 0 & 0 \\ 0 & i & 0 & 0 & 1 & 0 \\ -1+\rho & 0 & -3\rho^2 & 0 & 0 & 3 \end{pmatrix}$.

Analytic and rational representation of the automorphism α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & 0 & \frac{\rho-1}{3} \\ 0 & i & 0 \\ 0 & 0 & \rho \end{pmatrix} \text{ and } \rho_r(\alpha) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

$K \simeq \mathbb{Z}/6\mathbb{Z}$

No e): Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau-4\rho^2}{3} & 0 & 2-2\rho & 1 & 0 & 0 \\ 0 & 4\rho & 6\rho & 0 & 2 & 0 \\ 2-2\rho & 6\rho & 3+12\rho & 0 & 0 & 6 \end{pmatrix}$.

Analytic and rational representation of the automorphism α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & \rho-1 & 2\frac{1-\rho}{3} \\ 0 & -\rho^2 & 0 \\ 0 & \frac{3}{2} & \rho \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 2 & -3 \\ 2 & 0 & 0 & 0 & -1 & 2 \\ -4 & 0 & 6 & 1 & 0 & 0 \\ 0 & -2 & -3 & 0 & 1 & 0 \\ -1 & -1 & -2 & 0 & 1 & -1 \end{pmatrix}$$

$K \sim \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

No f): Let (X, α) be the pair of No g) below. Then $(X, -\alpha^{-1})$ is an example for No f) of Table 5.6.

No g): Periodmatrix: $\pi = \begin{pmatrix} \frac{\tau-5+3i+2\rho}{6} & -2+i+\rho & -5+3i+2\rho & 1 & 0 & 0 \\ -2+i+\rho & 3+2i+3\rho & 6+6i+6\rho & 0 & 1 & 0 \\ -5+3i+2\rho & 6+6i+6\rho & 12+18i+12\rho & 0 & 0 & 6 \end{pmatrix}$.

Analytic and rational representation of the automorphism α:

$$\rho_a(\alpha) = \begin{pmatrix} 1 & -i+\rho & \frac{1+3i+2\rho^2}{6} \\ 0 & -2i+3\rho & i-\rho \\ 0 & -6i+6\rho & 3i-2\rho \end{pmatrix} \text{ and } \rho_r(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 5 & -12 \\ 0 & 0 & 0 & 0 & -2 & 5 \\ -2 & -5 & -12 & 1 & 0 & 0 \\ 3 & -5 & -12 & 0 & -3 & 6 \\ 1 & -2 & -5 & 0 & -1 & 2 \end{pmatrix}$$

$K \simeq \mathbb{Z}/2\mathbb{Z}$

No h): Let (X, α) be the pair of No e). Then $(X, -\alpha)$ is an example for No h) of Table 5.6.

No i): Let (X, α) be a pair of No d). Then $(X, -\alpha)$ is an example for No i) of Table 5.6 .

No k): One can show (see **[SCH]**, Sec 5.35) that if (X, α) is as in No k) of Table 5.6, then necessarily K is trivial, that is $X \in E_0 \times E_{1728} \times E_0$ and $\alpha = \rho \times i \times (-\rho)$.

References

[CAV] Lange, H., Birkenhake, Ch.: *Complex Abelian Varieties*, Grundlehren 302, Springer Verlag (1992).

[FU] Fujiki, A.: *Finite Automorphism Groups of Complex Tori of Dimension Two*, Publ. RIMS, Kyoto Univ. 24, 1-97 (1988).

[BGL] Birkenhake, Ch., González-Aguilera, V., Lange, H.: *Automorphism Groups of 3-dimensional Complex Tori*, Preprint (1998).

[BL] Birkenhake, Ch., Lange, H.: *Cubic theta relations*, Journal fur die reine angew. Math 407, 167-177 (1990)

[MUM] Mumford, D.: *Abelian Varieties*, Bombay (1974).

[RO] Roan, S.: *Complex 3-Tori with an Order 7 Special Automorphism*, Preprint (1990).

[RAU] Rauch, H.E., Lewittes, J.: *The Riemann surface of Klein with 168 automorphisms*, Problems in Analysis, a symposium in honor of Solomon Bochner,

[RIE] Ries, X.: *The splitting of some Jacobi varieties using Automorphism Groups*, Contemp. Math. 102, 81-124 (1997)

[SCH] Schmidt,D.: *Automorphismen 2 und 3-dimensionaler abelscher Varietäten*, Dissertation, Universität Erlangen (1997)

Ch. Birkenhake, Mathematisches Institut, Bismarckstr. 1 1/2, D 91054 Erlangen
E-mail address: birken@mi.uni-erlangen.de

V. González-Aguilera, Departamento de Matemática. Universidad Santa María, Casilla 110-V. Valparaiso
E-mail address: vgonzale@mat.utfsm.cl

H. Lange, Mathematisches Institut, Bismarckstr. 1 1/2, D 91054 Erlangen
E-mail address: lange@mi.uni-erlangen.de